

Long-Range Correlations of the Surface Charge Density Between Electrical Media with Flat and Spherical Interfaces

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We study the asymptotic long-range behavior of the time-dependent correlation function of the surface charge density induced on the interface between two media of distinct dielectric functions which are in thermal equilibrium with one another as well as with the radiated electromagnetic field. We start with a short review which summarizes the results obtained by using quantum and classical descriptions of media, in both non-retarded and retarded regimes of particle interactions. The classical static result for the flat interface is rederived by using a combination of the microscopic linear response theory and the macroscopic method of electrostatic image charges. The method is then applied to the case of a spherically shaped interface between media.

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1 A short review

This article is about a simple inhomogeneous physical system consisting of two distinct electrical media in contact with one another which are in thermal equilibrium. Such system is easily accessible in experiments and can serve as a tool for testing quantum mechanics (and its classical limit) and crossover between retardation and non-retarded effects.

The special geometry of two semi-infinite media with a flat interface, formulated in the three-dimensional Cartesian space of points $\mathbf{r} = (x, y, z)$, is pictured in Fig. 1. The two media with the distinct frequency-dependent dielectric functions $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ occupy the complementary half-spaces $x > 0$ and $x < 0$, respectively. The flat interface between media is the plane $x = 0$. We recall that, in Gauss units, $\epsilon(\omega) = 1$ for vacuum, while the static dielectric constant $\epsilon(0)$ is finite (> 1) for dielectrics and diverging, $\epsilon(0) \rightarrow i\infty$, for conductors. The system is translationally invariant along each plane formed by the two coordinates $\mathbf{R} = (y, z)$ perpendicular to x . We assume that the media have no magnetic structure and the magnetic permeabilities $\mu_1 = \mu_2 = 1$. Non-relativistic charged particles forming the two media and the radiated electromagnetic (EM) field are in thermal equilibrium at some temperature T , or the inverse temperature $\beta = 1/(k_B T)$ with k_B being the Boltzmann constant.

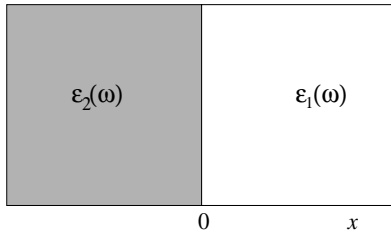


Fig. 1 The geometry of two semi-infinite electrical media with the flat interface contact at $x = 0$.

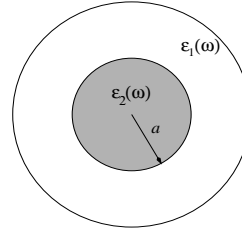


Fig. 2 The geometry of two electrical media with the spherical interface of radius a .

The different electric properties of the media give rise to a microscopic surface charge density (operator in quantum mechanics) $\sigma(t, \mathbf{R})$ at time t and point $\mathbf{r} = (0, \mathbf{R})$ on the interface. It is understood as being the

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microscopic volume charge density integrated along the x -axis on some microscopic depth from the interface. According to the elementary electrodynamics, the surface charge density is associated with the discontinuity of the normal x component of the microscopic electric field at the interface:

$$\sigma(t, \mathbf{R}) = \frac{1}{4\pi} [E_x^+(t, \mathbf{R}) - E_x^-(t, \mathbf{R})], \quad (1)$$

where the superscript $+/-$ means approaching the interface through the limit $x \rightarrow 0^+/0^-$. The tangential y and z components of the electric field are continuous at the interface. The correlation of fluctuations of the surface charge density at two points on the interface, with times different by t and distances different by $R = |\mathbf{R}|$, is described by the (symmetrized) correlation function

$$S(t, R) \equiv \frac{1}{2} \langle \sigma(t, \mathbf{R})\sigma(0, \mathbf{0}) + \sigma(0, \mathbf{0})\sigma(t, \mathbf{R}) \rangle^T, \quad (2)$$

where $\langle \dots \rangle^T$ represents a truncated equilibrium average at the inverse temperature β , $\langle AB \rangle^T = \langle AB \rangle - \langle A \rangle \langle B \rangle$. With regard to (1), this correlation function is related to fluctuations of the electric field on the interface.

Although the system is not in a critical state, the combination of the spatial inhomogeneity and long-ranged EM interactions in the media causes that the asymptotic large-distance behavior of the surface charge correlation function (2) exhibits, in general, the long-range tail of type

$$\beta S(t, R) \sim \frac{h(t)}{R^3}, \quad R \rightarrow \infty \quad (3)$$

with some prefactor (slope) function $h(t)$. It is useful to introduce the two-dimensional Fourier transform $S(t, q) = \int d^2R \exp(-i\mathbf{q} \cdot \mathbf{R}) S(t, \mathbf{R})$, where the wave vector $\mathbf{q} = (q_y, q_z)$. In the sense of distributions, the Fourier transform of $1/R^3$ is $-2\pi q$. Consequently, $\beta S(t, q)$ has a kink singularity at $\mathbf{q} = \mathbf{0}$: $\beta S(t, q) \sim -2\pi h(t)q$ for $q \rightarrow 0$. $S(t, q)$ is measurable by scattering experiments. The fact that it is linear in q for small q makes this quantity very different from the usual bulk structure functions with short-range (usually exponential) decays which are proportional to q^2 in the limit $q \rightarrow 0$.

The form of the prefactor function $h(t)$ depends on “the physical model” used. Defining media, one can apply:

- Either *classical* mechanics, $h_{\text{cl}}(t)$, or *quantum* mechanics, $h_{\text{qu}}(t)$. According to the correspondence principle, the quantum mechanics reduces to the classical one in the high-temperature limit.
- Either *non-retarded* description, $h^{(\text{nr})}$, or *retarded* description, $h^{(\text{r})}$, of particle interactions. In the non-retarded regime, the speed of light is taken to be infinite, $c = \infty$, ignoring in this way magnetic forces. In the retarded regime, c is assumed finite and so the charged particles are fully coupled to both electric and magnetic parts of the radiated EM field. According to the Bohr-Van Leeuwen theorem [1], magnetic degrees of freedom can be effectively eliminated from statistical averages of classical systems for the static case $t = 0$, so that $h_{\text{cl}}^{(\text{r})}(0) = h_{\text{cl}}^{(\text{nr})}(0) \equiv h_{\text{cl}}(0)$. If the EM field is thermalized, it was shown in Ref. [2] that the decoupling arises only at distances larger than the thermal photon wavelength $\beta\hbar c$, where the EM field itself can be treated classically. Quantum electrodynamics must be used at shorter distances, which invalidates the decoupling. At room temperature, the wavelength $\beta\hbar c$ is rather large compared to usual microscopic scales and might have some experimental importance. The decoupling is preserved in our macroscopic electrodynamics. For time differences $t \neq 0$, magnetic interactions do affect classical charged systems in thermal equilibrium and so, in general, $h_{\text{cl}}^{(\text{r})}(t) \neq h_{\text{cl}}^{(\text{nr})}(t)$.

The media configuration studied in the past was exclusively a conductor in contact with vacuum. When the conductor is modelled by a *classical* Coulomb fluid, a microscopic analysis [3] leads to the universal static result

$$h_{\text{cl}}(0) = -\frac{1}{8\pi^2} \quad (\text{conductor vs. vacuum}), \quad (4)$$

independent of the fluid composition. The same result has been obtained later [4] by using simple macroscopic arguments. As concerns *quantum* microscopic models of Coulomb fluids, a substantial simplification arises in the so-called jellium (one-component plasma), i.e. the system of identical pointlike particles of charge e , mass

m and bulk number density n , immersed in a fixed uniform neutralizing background of charge density $-en$. The dynamical properties of the jellium have a special feature: There is no viscous damping of the long-wavelength plasma oscillations [5]. In the non-retarded regime of the Maxwell equations, the frequencies of nondispersive long-wavelength collective modes, namely ω_p of the bulk plasmons and ω_s of the surface plasmons, are given by $\omega_p = \sqrt{4\pi ne^2/m}$ and $\omega_s = \omega_p/\sqrt{2}$. The dielectric function of the jellium is described by a one-resonance Drude formula

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\eta)}, \quad (5)$$

where the dissipation constant η is taken as positive infinitesimal, $\eta \rightarrow 0^+$. The obtained time-dependent result has the non-universal form [6]

$$h_{\text{qu}}^{(\text{nr})}(t) = -\frac{1}{8\pi^2} [2g(\omega_s) \cos(\omega_s t) - g(\omega_p) \cos(\omega_p t)] \quad (\text{jellium vs. vacuum}), \quad (6)$$

where the function

$$g(\omega) \equiv \frac{\beta\hbar\omega}{2} \coth\left(\frac{\beta\hbar\omega}{2}\right) = 1 + \sum_{j=1}^{\infty} \frac{2\omega^2}{\omega^2 + \xi_j^2}, \quad \xi_j = \frac{2\pi}{\beta\hbar} j. \quad (7)$$

In the complex upper half-plane, $g(\omega)$ possesses an infinite sequence of simple poles at the (imaginary) Matsubara frequencies $i\xi_j$ ($j = 1, 2, \dots$). In the classical limit $\beta\hbar\omega \rightarrow 0$, the function $g(\omega) \rightarrow 1$ for any ω and the quantum formula (6) reduces to

$$h_{\text{cl}}^{(\text{nr})}(t) = -\frac{1}{8\pi^2} [2 \cos(\omega_s t) - \cos(\omega_p t)]. \quad (8)$$

For $t = 0$, we recover the universal classical static result (4).

To study a general configuration of two media in contact and the effect of retardation, in a series of works [7, 8] we applied the macroscopic fluctuational electrodynamics of Rytov [9] to the present problem. The classical static result was found in the form

$$h_{\text{cl}}(0) = -\frac{1}{8\pi^2} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - \frac{4}{\epsilon_1 + \epsilon_2} \right), \quad (9)$$

where $\epsilon_1 \equiv \epsilon_1(0)$ and $\epsilon_2 \equiv \epsilon_2(0)$. For the previously studied configuration of a conductor ($\epsilon_1 \rightarrow i\infty$ and vacuum ($\epsilon_2 = 1$), we recover the universal formula (4). In the retarded regime, for any t we found that

$$h_{\text{qu}}^{(\text{r})}(t) = h_{\text{cl}}(0), \quad (10)$$

independently of \hbar , c and the temperature. This surprising result was verified explicitly on a microscopic model of jellium with retarded interactions [8]. In the non-retarded regime, we got

$$h_{\text{qu}}^{(\text{nr})}(t) = -\frac{1}{8\pi^2} \left[\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - \frac{4}{\epsilon_1 + \epsilon_2} + \text{Re } f(it) \right], \quad (11)$$

where $f(it)$ is an analytic $\tau \rightarrow it$ continuation of the function

$$f(\tau) = 2 \sum_{j=1}^{\infty} \left[\frac{1}{\epsilon_1(i\xi_j)} + \frac{1}{\epsilon_2(i\xi_j)} - \frac{4}{\epsilon_1(i\xi_j) + \epsilon_2(i\xi_j)} \right] \cos(\xi_j \tau). \quad (12)$$

Plugging into this expression the dielectric function of the jellium (5), one recovers the microscopic result (6). A crossover between the retarded and non-retarded regions was found at $q_{\text{cross}} \sim \omega_p/c$: If $q < q_{\text{cross}}$ ($q > q_{\text{cross}}$), the slope function $h^{(\text{r})}$ ($h^{(\text{nr})}$) takes place.

After the short overview, we shall investigate how the classical static result (9) modifies itself for a spherically curved interface between two media (see Fig. 2). Firstly, in section 2, we provide an alternative derivation of (9) for the flat interface by using a combination of microscopic and macroscopic approaches. This method allows us to generalize the classical static result to the spherical interface between media in section 3.

2 Rederivation of the classical static result (9) for the flat interface

Our strategy is to compute, for an arbitrary configuration of two points \mathbf{r} and \mathbf{r}' in the media 1 and 2, the correlation function $\langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle$, where $\phi(\mathbf{r})$ is the microscopic electric potential created by constituents of the media at point \mathbf{r} . It is related to the microscopic charge density ρ by $\phi(\mathbf{r}) = \int d\mathbf{r}'' \rho(\mathbf{r}'')/|\mathbf{r} - \mathbf{r}''|$, where the integral is over the whole space. Since $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$, we have

$$\langle E_x(\mathbf{r})E_x(\mathbf{r}') \rangle^T = \frac{\partial^2}{\partial x \partial x'} \langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle^T. \quad (13)$$

According to the relation (1), the surface charge correlation function is given by

$$\langle \sigma(\mathbf{R})\sigma(\mathbf{R}') \rangle^T = \frac{1}{(4\pi)^2} \langle E_x^+(\mathbf{R})E_x^+(\mathbf{R}') + E_x^-(\mathbf{R})E_x^-(\mathbf{R}') - E_x^+(\mathbf{R})E_x^-(\mathbf{R}') - E_x^-(\mathbf{R})E_x^+(\mathbf{R}') \rangle^T. \quad (14)$$

Let us introduce an infinitesimal test charge q at point \mathbf{r} . Microscopically, denoting by $\phi_{\text{tot}}(\mathbf{r}')$ the *total* (i.e. direct plus due to the particles) microscopic potential induced at point \mathbf{r}' , it holds $\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q = q/|\mathbf{r} - \mathbf{r}'| + \langle \phi(\mathbf{r}') \rangle_q$, where $\langle \dots \rangle_q$ means an equilibrium average in the presence of charge q . The additional Hamiltonian is $\delta H = q\phi(\mathbf{r})$. Using the classical linear response theory for $\langle \phi(\mathbf{r}') \rangle_q$, we obtain

$$\langle \phi(\mathbf{r}') \rangle_q = \langle \phi(\mathbf{r}') \rangle - \beta \langle \phi(\mathbf{r}') \delta H \rangle^T = \langle \phi(\mathbf{r}') \rangle - \beta q \langle \phi(\mathbf{r}') \phi(\mathbf{r}) \rangle^T, \quad (15)$$

where $\langle \dots \rangle$ means the standard equilibrium average in the absence of the test charge q . Consequently,

$$\beta \langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle^T = \frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{1}{q} [\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q - \langle \phi(\mathbf{r}') \rangle]. \quad (16)$$

The shift of the mean potential at \mathbf{r}' due to the test charge q at \mathbf{r} , $\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q - \langle \phi(\mathbf{r}') \rangle$, is determined by using the macroscopic method of images [10, 11]. If both points \mathbf{r} and \mathbf{r}' are inside medium 1, the shift is given by

$$\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q - \langle \phi(\mathbf{r}') \rangle = \frac{q}{\epsilon_1 |\mathbf{r}' - \mathbf{r}|} + \frac{q'}{\epsilon_1 |\mathbf{r}' - \mathbf{r}^*|}, \quad q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q, \quad (17)$$

where $\mathbf{r}^* = (-x, \mathbf{R})$ is the position of the image charge q' . If both points \mathbf{r} and \mathbf{r}' are inside medium 2, the formula (17) with interchanged indices 1 and 2 applies. If \mathbf{r} is in medium 1 and \mathbf{r}' is in medium 2, or vice versa, it holds

$$\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q - \langle \phi(\mathbf{r}') \rangle = \frac{q''}{\epsilon_2 |\mathbf{r}' - \mathbf{r}|}, \quad q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q. \quad (18)$$

Inserting the shift of the mean potential to (16) for all four possible configurations of points \mathbf{r} and \mathbf{r}' in media 1 and 2, using (13) and the relations

$$\left. \frac{\partial^2}{\partial x \partial x'} \frac{1}{|\mathbf{r}' - \mathbf{r}|} \right|_{x=x'=0} = \frac{1}{|\mathbf{R} - \mathbf{R}'|^3}, \quad \left. \frac{\partial^2}{\partial x \partial x'} \frac{1}{|\mathbf{r}' - \mathbf{r}^*|} \right|_{x=x'=0} = -\frac{1}{|\mathbf{R} - \mathbf{R}'|^3}, \quad (19)$$

we find that

$$\beta \langle E_x^\pm(\mathbf{R})E_x^\pm(\mathbf{R}') \rangle^T = \left(1 - \frac{2}{\epsilon_{1,2}} + \frac{2}{\epsilon_1 + \epsilon_2} \right) \frac{1}{|\mathbf{R} - \mathbf{R}'|^3} \quad (20)$$

$$\beta \langle E_x^\pm(\mathbf{R})E_x^\mp(\mathbf{R}') \rangle^T = \left(1 - \frac{2}{\epsilon_1 + \epsilon_2} \right) \frac{1}{|\mathbf{R} - \mathbf{R}'|^3}. \quad (21)$$

Considering these correlators in (14), we end up with the classical static result (9).

3 Classical static result for a spherical interface between media

The same method is now applied to the spherical interface of radius a between media 1 ($r > a$) and 2 ($r < a$), see Fig. 2. The microscopic formula (16) remains unchanged. The macroscopic shift of the mean potential at \mathbf{r}' due to the test charge q at \mathbf{r} for the spherical geometry can be expanded in terms of the Legendre polynomials $\{P_l(\cos \theta)\}_{l=0}^{\infty}$, with θ being the angle between \mathbf{r} and \mathbf{r}' , as [11]

$$\frac{1}{q} [\langle \phi_{\text{tot}}(\mathbf{r}') \rangle_q - \langle \phi(\mathbf{r}') \rangle] = \sum_{l=0}^{\infty} g_l(r, r') P_l(\cos \theta). \quad (22)$$

Denoting by $r_{<}$ ($r_{>}$) the smaller (larger) of $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$, one has

$$g_l(r, r') = \frac{1}{\epsilon_1} \left[r_{<}^l + \frac{(\epsilon_1 - \epsilon_2)l}{\epsilon_1(l+1) + \epsilon_2 l} \frac{a^{2l+1}}{r_{<}^{l+1}} \right] \frac{1}{r_{>}^{l+1}} \quad \text{for } r > a \text{ and } r' > a, \quad (23)$$

$$g_l(r, r') = \frac{1}{\epsilon_2} r_{<}^l \left[\frac{1}{r_{>}^{l+1}} + \frac{(\epsilon_2 - \epsilon_1)(l+1)}{\epsilon_1(l+1) + \epsilon_2 l} \frac{r_{>}^l}{a^{2l+1}} \right] \quad \text{for } r < a \text{ and } r' < a, \quad (24)$$

$$g_l(r, r') = \frac{2l+1}{\epsilon_1(l+1) + \epsilon_2 l} \frac{r'^l}{r^{l+1}} \quad \text{for } r > a \text{ and } r' < a. \quad (25)$$

We define the image position of a charge at \mathbf{r} as $\mathbf{r}^* = (a/r)^2 \mathbf{r}$ and consider the summation formulas [12]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} = \sum_{l=0}^{\infty} P_l(\cos \theta) \frac{r_{<}^l}{r_{>}^{l+1}}, \quad (26)$$

$$\frac{a}{r} \frac{1}{|\mathbf{r}^* - \mathbf{r}'|} = \frac{1}{\sqrt{a^2 + (rr')^2/a^2 - 2rr' \cos \theta}} = \begin{cases} \sum_{l=0}^{\infty} P_l(\cos \theta) \frac{a^{2l+1}}{(rr')^{l+1}} & \text{if } r > a \text{ and } r' > a, \\ \sum_{l=0}^{\infty} P_l(\cos \theta) \frac{(rr')^l}{a^{2l+1}} & \text{if } r < a \text{ and } r' < a, \end{cases} \quad (27)$$

$$\sum_{l=0}^{\infty} P_l(\cos \theta) \frac{t^l}{l + \alpha} = \int_0^1 dx \frac{x^{\alpha-1}}{\sqrt{1 + (tx)^2 - 2tx \cos \theta}} \quad \text{for } |t| < 1 \text{ and } \alpha > 0, \quad (28)$$

to obtain, using (16), the two-point electric potential correlation functions:

$$\begin{aligned} \beta \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle^T &= \left(1 - \frac{1}{\epsilon_1}\right) \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1(\epsilon_1 + \epsilon_2)} \frac{a}{r} \frac{1}{|\mathbf{r}^* - \mathbf{r}'|} \\ &+ \frac{\epsilon_1 - \epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \int_0^1 dx \frac{x^{-\nu}}{\sqrt{a^2 x^2 + (rr')^2/a^2 - 2rr' x \cos \theta}} \end{aligned} \quad (29)$$

for $r > a$ and $r' > a$,

$$\begin{aligned} \beta \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle^T &= \left(1 - \frac{1}{\epsilon_2}\right) \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2(\epsilon_1 + \epsilon_2)} \frac{a}{r} \frac{1}{|\mathbf{r}^* - \mathbf{r}'|} \\ &+ \frac{\epsilon_1 - \epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \int_0^1 dx \frac{x^{-\nu}}{\sqrt{a^2 + (rr'x)^2/a^2 - 2rr'x \cos \theta}} \end{aligned} \quad (30)$$

for $r < a$ and $r' < a$,

$$\beta \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle^T = \left(1 - \frac{2}{\epsilon_1 + \epsilon_2}\right) \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \frac{\epsilon_1 - \epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \int_0^1 dx \frac{x^{-\nu}}{\sqrt{r^2 + (r'x)^2 - 2rr'x \cos \theta}} \quad (31)$$

for $r > a$ and $r' < a$. Here, the exponent $\nu = \epsilon_2/(\epsilon_1 + \epsilon_2)$ lies in the interval $[0, 1]$.

The radial component of the electric field, normal to the surface of the sphere, is given by $E_n(\mathbf{r}) = -\partial_r \phi(\mathbf{r})$. Thus the counterpart of (13) reads $\langle E_n(\mathbf{r}) E_n(\mathbf{r}') \rangle^T = \partial_{rr'}^2 \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle^T$. Using the relation (14), with E_x^\pm substituted by E_n^\pm , after lengthy algebra we find the surface charge correlation function between two points \mathbf{R} and \mathbf{R}' on the sphere interface of the form

$$\beta \langle \sigma(\mathbf{R}) \sigma(\mathbf{R}') \rangle^T = -\frac{1}{8\pi^2} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - \frac{4}{\epsilon_1 + \epsilon_2} \right) \frac{1}{|\mathbf{R} - \mathbf{R}'|^3} + \frac{1}{16\pi^2} \frac{\epsilon_1 - \epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \frac{1}{a^3} I(\nu, \cos \theta), \quad (32)$$

where the integral

$$I(\nu, \cos \theta) = \int_0^1 dx x^{-\nu} \left[\frac{3(1-x^2)^2}{(1+x^2-2x\cos\theta)^{5/2}} - \frac{2(1+x^2)}{(1+x^2-2x\cos\theta)^{3/2}} \right]. \quad (33)$$

Here, the angle θ between the points \mathbf{R} and \mathbf{R}' on the sphere of radius a is given by $\cos \theta = 1 - \frac{1}{2}(|\mathbf{R} - \mathbf{R}'|/a)^2$. The above two formulas hold for all macroscopic distances $|\mathbf{R} - \mathbf{R}'|$ and sphere radii a .

A special case of physical interest is the fixed distance $|\mathbf{R} - \mathbf{R}'|$ and the large radius limit $|\mathbf{R} - \mathbf{R}'|/a \rightarrow 0$. Writing $1 + x^2 - 2x\cos\theta = (1-x)^2 + \varepsilon x$ with $\varepsilon = (|\mathbf{R} - \mathbf{R}'|/a)^2 \ll 1$ and using the small- ε expansions

$$\int_0^1 dx \frac{x^\mu}{[(1-x)^2 + \varepsilon x]^{5/2}} = \frac{2}{3\varepsilon^2} + \frac{3-2\mu}{6\varepsilon^{3/2}} + \frac{2-3\mu+\mu^2}{6\varepsilon} + \dots, \quad (34)$$

$$\int_0^1 dx \frac{x^\mu}{[(1-x)^2 + \varepsilon x]^{3/2}} = \frac{1}{\varepsilon} + \dots \quad (35)$$

($\mu > -1$), we obtain $I(\nu, \cos \theta) = 2/\varepsilon + \mathcal{O}(1)$. Thus from (32) we conclude that

$$\begin{aligned} \beta \langle \sigma(\mathbf{R}) \sigma(\mathbf{R}') \rangle^T &= -\frac{1}{8\pi^2} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - \frac{4}{\epsilon_1 + \epsilon_2} \right) \frac{1}{|\mathbf{R} - \mathbf{R}'|^3} \\ &\quad + \frac{1}{8\pi^2} \frac{\epsilon_1 - \epsilon_2}{(\epsilon_1 + \epsilon_2)^2} \frac{1}{|\mathbf{R} - \mathbf{R}'|^2} \frac{1}{a} + \mathcal{O}\left(\frac{1}{a^3}\right). \end{aligned} \quad (36)$$

The leading term, invariant with respect to the media interchange $\epsilon_1 \leftrightarrow \epsilon_2$, corresponds to the classical static result for the flat interface (9). The first curvature correction changes the sign under $\epsilon_1 \leftrightarrow \epsilon_2$; this is expected since the media interchange is effectively equivalent to the change of the interface curvature to the opposite one.

Another interesting case is the antipode configuration with $|\mathbf{R} - \mathbf{R}'| = 2a$, for which curvature effects are not erased. Substituting $\cos \theta = -1$ in the integral (33), $I(\nu, -1)$ is expressible in terms of harmonic numbers [12].

An open question is a sphere extension of the quantum time-dependent result (10), derived for a flat interface by taking into account the retardation. This result should be recovered in the large radius limit $|\mathbf{R} - \mathbf{R}'| \ll a$, while curvature effects might invalidate it for $|\mathbf{R} - \mathbf{R}'| \sim a$. The answer requires the application of Rytov's EM fluctuational theory [9] to curved interfaces between media which is a difficult task left for near future.

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References

- [1] N. Bohr, Dissertation (Copenhagen, 1911); H.-J. Van Leeuwen, J. Phys. Radium **2**, 361 (1921).
- [2] A. Alastuey and W. Appel, Physica A **276**, 508 (2000).
- [3] B. Jancovici, J. Stat. Phys. **29**, 263 (1982).
- [4] B. Jancovici, J. Stat. Phys. **80**, 445 (1995).
- [5] D. Pines and P. Nozières, The Theory of Quantum Liquids (Benjamin, New York, 1966).
- [6] B. Jancovici, J. Stat. Phys. **39**, 427 (1985); B. Jancovici, J. L. Lebowitz, and Ph. A. Martin, J. Stat. Phys. **41**, 941 (1985).
- [7] L. Šamaj and B. Jancovici, Phys. Rev. E **78**, 051119 (2008).
- [8] B. Jancovici and L. Šamaj, Phys. Rev. E **79**, 021111 (2009), Phys. Rev. E **80**, 031139 (2009).
- [9] S. M. Rytov, Sov. JETP **6**, 130 (1958); M. L. Levin and S. M. Rytov, Theory of Equilibrium Thermal Fluctuations in Electrodynamics (Science, Moscow, 1967); E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Part 2 (Pergamon, Oxford, 1981).
- [10] J. D. Jackson, Classical Electrodynamics, 3rd edn. (Wiley, New York, 1999).
- [11] J. Schwinger, L. L. DeRaad, K. A. Milton, and W.-Y. Tsai, Classical Electrodynamics (Perseus, Cambridge, 1998).
- [12] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 6th edn. (Academic, London, 2000).